IB GROUPS, RINGS AND MODULES

Lent Term 2022
Example Sheet 4 of 4 **Rong Zhou**

All rings in this course are commutative with a 1.

(1) Let M be a module over a ring R, and let N be a submodule of M.

(a) Show that if N and M/N are finitely generated then so is M.
(b) Show that if M/N is free, then $M \cong N \oplus M/N$.

(2) We say that an R-module satisfies condition (N) if any submodule is finitely generated. Show that this condition is equivalent to condition (ACC): every increasing chain of submodules terminates.

Let R be a Noetherian ring. Show that the R-module R^n satisfies condition (N), and hence that any finitely generated R-module satisfies condition (N).

(3) Let M be a module over an integral domain R. We say that $m \in M$ is a *torsion element* if $rm = 0$ for some non-zero $r \in R$.

(a) Show that the set T of all torsion elements in M is a submodule of M, and that the quotient M/T is *torsion-free*—that is, contains no non-zero torsion elements.
(b) What are the torsion elements in the $\mathbb{Z}$-module $\mathbb{Q}/\mathbb{Z}$? In $\mathbb{R}/\mathbb{Z}$? In $\mathbb{R}/\mathbb{Q}$?
(c) Is the $\mathbb{Z}$-module $\mathbb{Q}$ torsion-free? Is it free? Is it finitely generated?

(4) Use elementary operations to put the integer matrix $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$ into Smith normal form D. Check your result using minors. Explain how to find invertible matrices P, Q for which $D = QAP$.

(5) Work out the Smith normal form of the matrices over $\mathbb{R}[X]$:

$$\begin{pmatrix} 2X-1 & X & X-1 & 1 \\ X & 0 & 1 & 0 \\ 0 & 1 & X & X \\ 1 & X^2 & 0 & 2X-2 \end{pmatrix} \text{ and } \begin{pmatrix} X^2+2X & 0 & 0 & 0 \\ 0 & X^2+3X+2 & 0 & 0 \\ 0 & 0 & X^3+2X^2 & 0 \\ 0 & 0 & 0 & X^4+X^3 \end{pmatrix}$$

(6) How many abelian groups are there of order 6? Of order 60? Of order 6000?

(7) Let G be the abelian group with generators a, b, c, and relations $6a + 10b = 0$, $6a + 15c = 0$, $10b + 15c = 0$. (That is, G is the free abelian group on generators a, b, c quotiented by the subgroup generated by the elements $6a+10b$, $6a+15c$, $10b+15c$.) Determine the structure of G as a direct sum of cyclic groups.

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(8) Prove that a finitely generated abelian group G is finite if and only if $G/pG = 0$ for some prime p. Give an example of a non-trivial abelian group G such that $G/pG = 0$ for all primes p.

(9) Let A be a complex matrix with characteristic polynomial $(X+1)^6(X-2)^3$ and minimal polynomial $(X+1)^3(X-2)^2$. Write down the possible Jordan normal forms for A. What are the invariant factors of the corresponding $\mathbb{C}[X]$-modules?

(10) Find a 2×2 matrix over $\mathbb{Z}[X]$ that is not equivalent to a diagonal matrix.

(11) Let M be a finitely generated module over a Noetherian ring R, and let f be an R-module homomorphism from M to itself. Does f injective imply f surjective? Does f surjective imply f injective? What happens if R is not Noetherian?

Further Questions

(12) A real $n \times n$ matrix A satisfies the equation $A^2 + I = 0$. Show that n is even and A is similar to a block matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ with each block an $m \times m$ matrix (where $n = 2m$).

(13) Show that a complex number α is an algebraic integer if and only if the additive group of the ring $\mathbb{Z}[\alpha]$ is finitely generated (i.e. $\mathbb{Z}[\alpha]$ is a finitely generated $\mathbb{Z}$-module). Furthermore if α and β are algebraic integers show that the subring $\mathbb{Z}[\alpha, \beta]$ of $\mathbb{C}$ generated by α and β also has a finitely generated additive group and deduce that $\alpha - \beta$ and $\alpha\beta$ are algebraic integers.

Show that the algebraic integers form a subring of $\mathbb{C}$.

(14) Show that the ring $C^\infty([-1,1])$ of all infinitely differentiable functions $[-1,1] \to \mathbb{R}$ (with pointwise operations) is not Noetherian.

(15) What is the rational canonical form of a matrix?

Show that the group $\mathrm{GL}_2(\mathbb{F}_2)$ of non-singular 2×2 matrices over the field $\mathbb{F}_2$ of 2 elements has three conjugacy classes of elements.

Show that the group $\mathrm{GL}_3(\mathbb{F}_2)$ of non-singular 3×3 matrices over $\mathbb{F}_2$ has six conjugacy classes of elements, corresponding to minimal polynomials $X+1$, $(X+1)^2$, $(X+1)^3$, X^3+1, X^3+X^2+1, X^3+X+1, one each of elements of orders $1, 2, 3$ and 4, and two of elements of order 7.

(16) Let $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2+\omega+1) = \{0, 1, \omega, \omega+1\}$, a field with four elements.

Show that the group $\mathrm{SL}_2(\mathbb{F}_4)$ of 2×2 matrices of determinant 1 over $\mathbb{F}_4$ has five conjugacy classes of elements, corresponding to minimal polynomials $X+1$, $(X+1)^2$, $(X+\omega)(X+\omega^2)$, $X^2+\omega X+1$ and $X^2+\omega^2 X+1$.

Show that the corresponding elements have orders $1, 2, 3, 5$ and 5, respectively.

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